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Piecewise Polynomial Methods of Haar Transform in Digital Signal Processing

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Abstract: This paper analyzes the mathematical foundations and practical significance of piecewise polynomial methods based on the Haar orthogonal basis in the process of digital signal processing. Algorithms for calculating spectral coefficients in Haar, Schauder, and spline bases are compared, and their structural and computational efficiency is presented through graphs and formulas. In particular, the advantages of fast transform algorithms adapted for piecewise-constant, piecewise-linear, and piecewise-quadratic bases are demonstrated, along with the challenges encountered during their implementation and possible solutions. It is shown that piecewise polynomial methods of the Haar transform can be effectively applied in signal processing systems that require high accuracy and speed.

Keywords: Haar transform, spectral coefficients, Schauder basis, parabolic spline, piecewise polynomial methods, digital signal.

Introduction: Digital signal processing has become an integral part of modern information technologies and electronics. One of the pressing challenges in this field is the identification of spectral characteristics within a signal and their efficient processing. In systems that require rapid processing, it is particularly important to reduce the number of computations, optimize memory usage, and maintain algorithmic simplicity [1,2]. From this perspective, algorithms based on the Haar orthogonal basis occupy a prominent position. In particular, the piecewise polynomial variants of the Haar transform — including piecewise-constant, piecewise-linear, and piecewise-quadratic bases — enable the analysis of signals in a segmented manner.

METHODOLOGY

One of the key features of Haar bases is the availability of fast algorithms for determining spectral coefficients. Fast algorithms enable a reduction in the number of arithmetic operations and memory usage during digital signal processing. As a result, the use of orthogonal bases in digital signal processing leads to an increase in processing speed. Fast Haar transform algorithms are widely applied for digital signal processing tasks [1,3].

The discrete Fourier-Haar series can be expressed in the following form:

$$f(x_n) = \sum_{i=0}^{n} C_i har_i(x_n) \tag{1}$$

These discrete transformations are derived by adjusting the C_n coefficients relative to the $har_i(x_n)$ Haar basis. The spectral coefficients are calculated using the following formulas:

$$C_0 = \frac{1}{N}\sum_{i=0}^{N-1} X(i)\,,\; C_n = \frac{1}{N}\sum_{i=0}^{N-1} X(i) har_n(i) \tag{2}$$

Fast algorithms are available for the efficient computation of Haar transforms, and they are referred to as the Fast Haar Transform (FHT) algorithms.

Fast Haar transform algorithm

In the computation of the discrete Haar transform, $N\log_2 N$ algebraic addition operations are performed. In the Fast Haar Transform (FHT), the number of required algebraic operations is $2(N-1)$ [1, 3, 6, 7]. This is approximately $0,5\log_2 N$ times fewer than the number of operations required in the standard discrete Haar transform algorithm. Therefore, in many practical applications, the use of the FHT algorithm for digital signal processing leads to a significant improvement in efficiency.

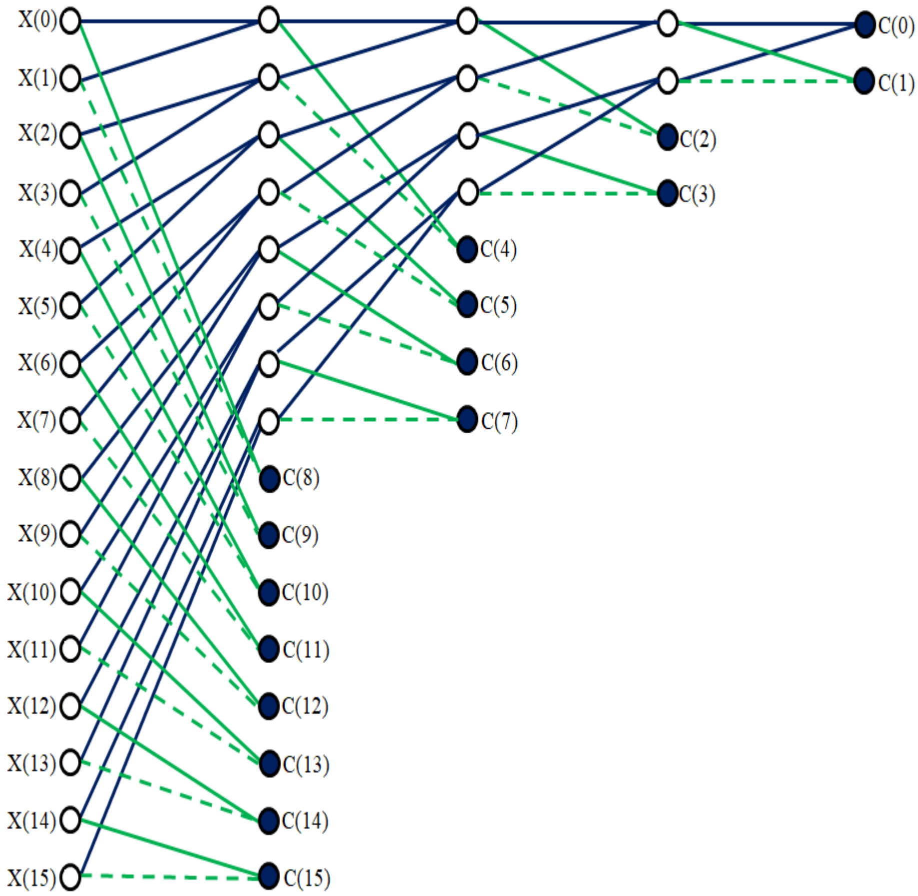

Fig. 1. Fast Haar Transform graph as proposed by Andrews

The fast algorithms for computing Haar coefficients are well-suited to the main characteristics of digital signal processors, as they primarily involve addition and subtraction operations. Representing the flow of signals and the sequence of computations in the form of a graph is an effective method for illustrating these processes. Figure 1 shows the graph of the FHT algorithm proposed by Andrews [3,6,7]. In these graphs, the number of input values is $N = 16$.

In the presented graph, continuous solid lines represent addition operations, whereas dashed lines correspond to subtraction operations. The input signals are indicated by $X(0), X(1), \dots, X(N-1)$, and the resulting Haar spectral coefficients are represented by $C(0), C(1), \dots, C(N-1)$.

Based on the FHT graph proposed by Andrews mentioned above, the block diagram of the algorithm for computing the spectral coefficients is illustrated in Figure 2.

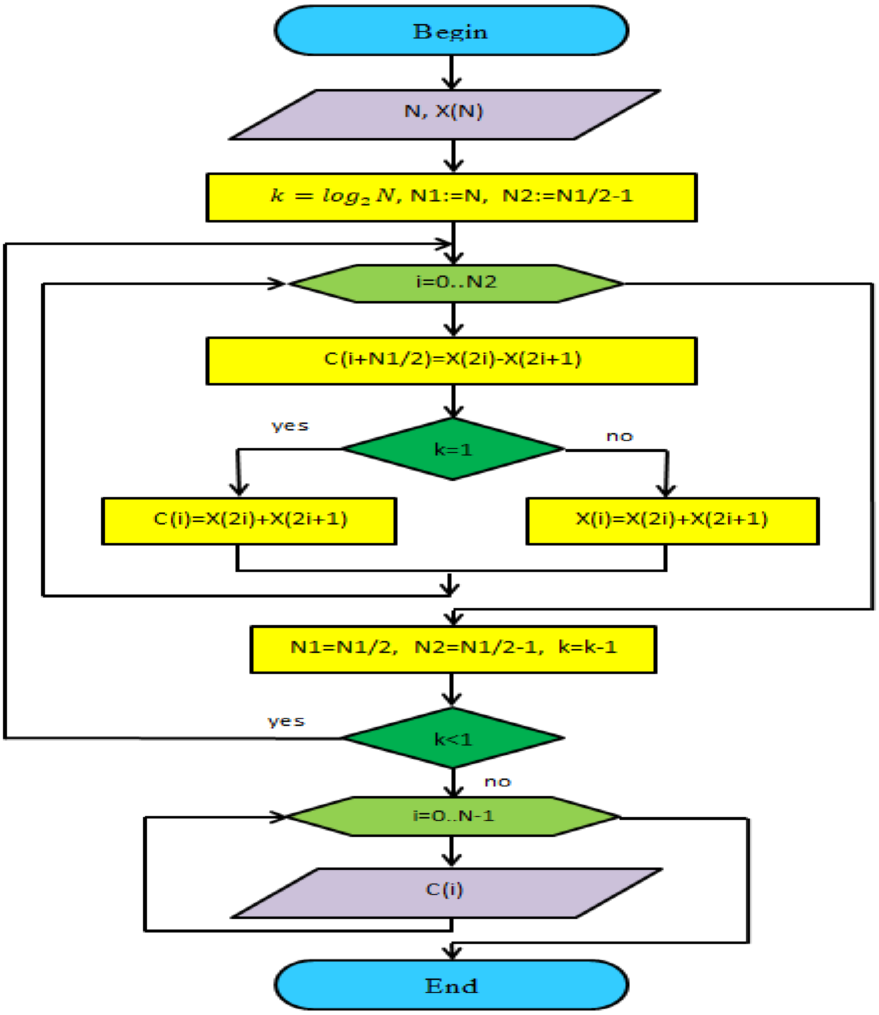

Fig. 2. Block diagram of the fast Haar transform algorithm

The key requirements for an efficient fast spectral transform algorithm are the minimization of the number of operations, the simplicity of each individual operation, and the reduction of memory consumption.

Using the sources provided in the literature [1–6, 8], we compare fast transform algorithms based on the number of addition and subtraction operations required for the computation of spectral coefficients.

In the Walsh Fast Transform algorithm, spectral processing of an array with $N = 2^k$ elements requires $k \cdot 2^k$ addition-subtraction operations. In the Fast Haar Transform algorithm, the number of required operations is $2(N-1)$, while in the Hartley Fast Transform algorithm, $3N-4$ operations are required. Here, N denotes the number of array elements, and k represents the number of iterations.

Table 1.

Comparison table of the number of arithmetic operations required for fast transform algorithms in one-dimensional digital signal processing.

Array size	Fast Transform (FT) algorithm name			
	Fast Fourier Transform	Fast Walsh Transform	Fast Haar Transform	Fast Harmut Transform

N	kN	kN	2(N-1)	3N-4
1024	10240	10240	2046	3068

Table 1 presents a comparison of the number of arithmetic operations required for the computation in the fast transform algorithms of Fourier, Walsh, Haar, and Hartley transforms. According to the data provided in the table, when the number of input values is 1024, the Fourier and Walsh fast transforms require 10,240 arithmetic operations, the fast Haar transform requires 2,046 operations, and the Fast Harmut Transform requires 3,068 operations. Thus, the fast Haar transform algorithm performs approximately 1.5 times fewer operations than the Hartley fast transform and about 5 times fewer operations compared to the Fourier and Walsh fast transform algorithms.

Piecewise polynomial methods of haar transform

In this section, we analyze the feasibility of applying fast transform algorithms, originally developed for orthogonal piecewise-constant basis functions, to the computation of coefficients in piecewise-linear bases. When expressed using integral representations, the Fourier-Haar formulas take the following form:

$$C_0 = \int_0^1 x(r)dr \cong \sum_{i=0}^{N-1} \int_{hpj} x(r)dr$$

$$C_k = \int_0^1 x(r)\cdot har_k(r)dr = \sum_{i=0}^{N-1} har(i)\int_{hpj} x(r)dr \qquad (3)$$

$$i = 1, 2, ..., n; \quad j = 0, 1, 2, ..., 2^{p-1}$$

This holds true only if the transformed signals $x(r)$ belong to the $L_2\left[0,1\right)$ metric space.

In this case, the properties of the local basis functions can be utilized, namely, among the 2^p functions of the same order, there is only one function h_{pj} that is nonzero within a given dyadic-rational interval

The computation algorithm for the coefficients presented in expression (3) does not exhibit the characteristics of a fast transform. Nevertheless, when it is necessary to determine spectral coefficients within local bases, the finite difference method can be applied directly.

For example, in the Schauder basis [6], the coefficients are calculated based on the following transformations:

$$\Delta f_i = \sum_{k=0}^{N-1} C_k \cdot Shd_k(X_{i+1}) - \sum_{k=0}^{N-1} C_k \cdot Shd_k(X_i) =$$

$$\sum_{k=0}^{N-1} C_k \cdot \left(\int Shd_k(r)dr - \int_{k=0} Shd_k(r)dr \right) = \frac{1}{2}\sum_{k=0}^{N-1} C_k har_k(x_i) \qquad (4)$$

$$C_k = \sum_{k=0}^{N-1} \Delta f_i har_k(x_i) \qquad (5)$$

Here, the Schauder fast transform algorithm based on the 'time-wise' reduction principles is presented. In expression (5), the sum on the right-hand side is divided into two parts: the first part includes the Δf_i differences with indices ranging from i=0 to i=n/2−1, and the second part includes the Δf_i differences with indices ranging from i=n/2 to i=n−1:

$$C_k = \sum_{i=0}^{N/2-1} \Delta f_i har_k(x_i) + \sum_{i=N/2}^{N-1} \Delta f_i har_k(x_i) \qquad (6)$$

The coefficients comprising the second half of the resulting vector, beginning from index n/2, can be computed during the first iteration. The sequence {Δf_i} is partitioned into n/4 groups, each consisting of two adjacent elements, Δf_{2i} and Δf_{2i+1}, to which the discrete Haar transform formula is subsequently applied for each pair:

$$C_{N/2+j} = \Delta f_{2j} - \Delta f_{2j+1}, \qquad j = 0,1,...,N/2-1$$

In the second iteration, the sequence is divided into groups consisting of 8 elements; in the fourth iteration, into groups of 16 elements, and so on. The corresponding values are determined according to the following formulas:

$$C_k = \sum_{i=0}^{N/2^p-1} \Delta f_i - \sum_{i=N/2^p}^{N/2^p-1} \Delta f_i \qquad (7)$$

The coefficients C_1 and C_0 in the final iteration are determined using the following formula:

$$C_1 = \sum_{i=0}^{N/2-1} \Delta f_i - \sum_{i=N/2}^{N-1} \Delta f_i$$

$$C_0 = \sum_{i=0}^{N-1} \Delta f_i$$

The graph representation of the Schauder fast transform algorithm for N=16 input values is shown in Figure 3.

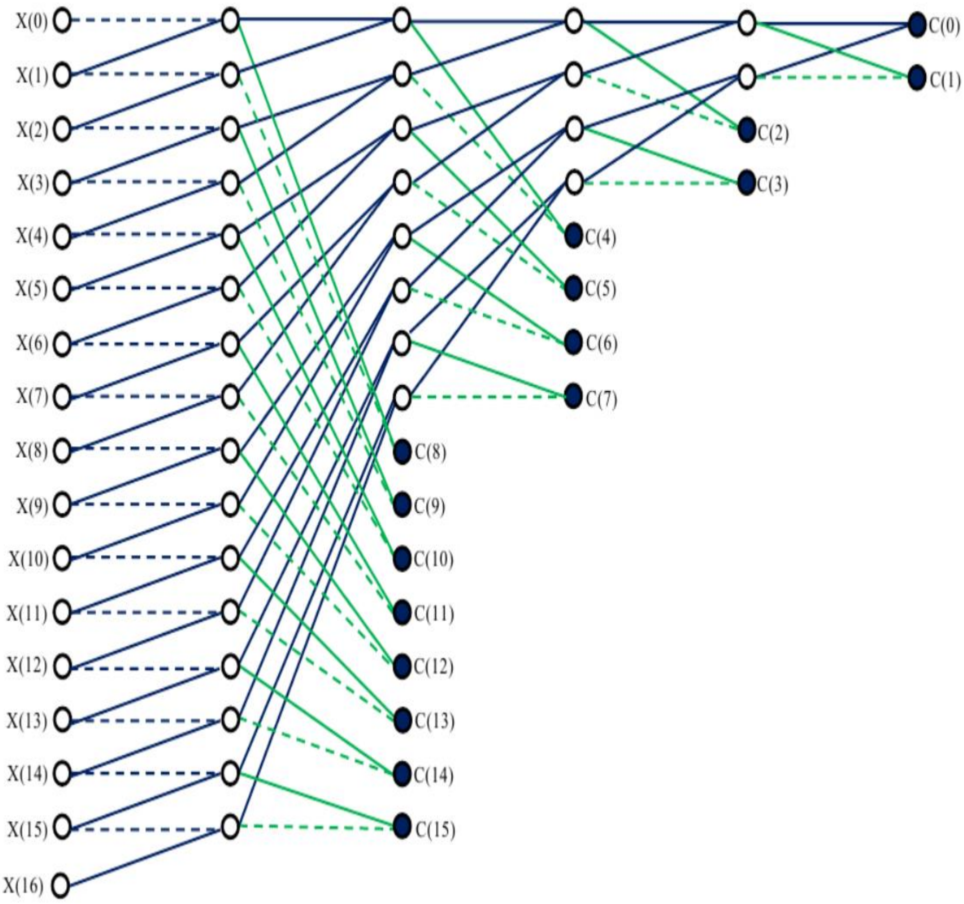

Fig. 3. Fast Transform Graph Based on the Schauder Basis

Thus, this algorithm enables the use of fast transform graphs based on orthogonal piecewise-constant basis functions for computing coefficients in piecewise-linear bases. A limitation of this algorithm is the difficulty that arises when dealing with small values of the Δf_i finite differences.

The analysis of coefficient computation methods across different bases has shown that such methods are available only for piecewise-constant and piecewise-linear bases. For piecewise-quadratic bases, coefficient computation methods have not yet been developed.

In fast transform algorithms utilizing local bases, the number of coefficients within groups of the same order is characterized by an exponential decrease as the number of iterations increases. This property can be exploited in the development of specialized high-speed computational architectures.

We proceed to examine a convergent series constructed from piecewise-quadratic Haar functions.

$$f(x) \cong \sum_{k=0}^{N-1} C_k haid_k(x) \qquad (8)$$

A drawback of the given series is the absence of fast algorithms for computing the C_k coefficients. This limitation can be overcome using a parabolic spline. If the second derivative of an interpolating parabolic spline of the function $f(x)$ over the interval [0,1] is taken, the result is a piecewise-constant function with step changes at the spline nodes, which can be expanded into a series based on the piecewise-constant orthogonal basis functions. As an example, we consider the expansion of the spline derivative into the Haar series:

$$S_2''(x) \cong \sum_{k=0}^{N-1} C_k har_k(x) \qquad (9)$$

According to the theorems concerning finite convergence and the integration of closed systems, the integration of both components leads to the following outcome:

$$S_2'(x) = 2^p \int_0^x S_2''(u)du = \sum_{k=0}^{N-1} C_k hain_k(x) + S_2'(0)$$

$$S_2(x) = \int_0^x S_2'(u)du + S_2(0) = \sum_{k=0}^{N-1} C_k haid_k(x) + S_2'(0) + S_2(0)$$

Consequently, at dyadic-rational interpolation nodes, the second derivative of the parabolic spline corresponds to the expansion of the interpolated function into a series of coefficients with respect to the Haar orthogonal basis functions. The first derivative of the spline corresponds to the expansion with respect to the $hain$ basis functions, while the spline itself corresponds to the expansion with respect to the $haid$ basis functions. The coefficients are determined from the linear component of the spline expansion, specifically, the first derivative evaluated at x=0 defines the linear term, whereas the constant component is given by the value of the spline $S_2(x)$ at the same point.

One of the most important properties of spline functions is the existence of higher-order derivatives. This feature enables the development of a hardware-oriented algorithm for computing coefficients within piecewise-parabolic bases, consisting of the following steps:

1. Input of the initial functional relationship, i.e., entering the set of real experimental data.
2. The b-coefficients are calculated according to the following formula:

$$b_i = \frac{1}{8}(-f_{i-1} + 10 f_i - f_{i+1}) \qquad (10)$$

The values approximating the spline $S_2(x)$ are determined according to the following formula:

$$f(x) \cong S_2(x) = b_{-1} \cdot B_{-1}(x) + b_0 \cdot B_0(x) + b_1 B_1(x) \qquad (11)$$

In this case, second-order derivatives are employed in place of the basis function values: $B''(x) = 1$ on the interval [−1.5,−0.5], $B''(x) = -2$ on [−0.5,0.5], and $B''(x) = 1$ on [0.5,1.5].

$$f''(x) \cong S_2(x) = b_{-1} \cdot B_{-1}''(x) + b_0 \cdot B_0''(x) + b_1 B_1''(x) \qquad (12)$$

As a result of these calculations, an array of values approximating the second derivative of the spline $S_2''(x_i)$ is obtained.

3. Formation of the $S_2''(x_i)$ array
4. Fast transforms are performed on the elements of the obtained array using a fast transform graph, and the coefficients are determined. These coefficients constitute the piecewise-quadratic basis coefficients.

The spectral coefficients are determined from the elements of the $S_2''(x_i)$ array using the fast transform algorithms presented above. These coefficients correspond to the coefficients in the piecewise-quadratic basis.

RESULTS

To assess the potential of digital signal processing based on Haar piecewise-polynomial bases, an investigation was carried out involving both an analytical function and geophysical signal data derived from experimental observations.

Table 2.

Research Results on Digital Signal Processing Using Haar Piecewise-Polynomial Methods

Type of function	N	Piecewise-constant Haar basis		Piecewise-linear Haar basis		Piecewise-quadratic Haar basis	
		$N_0\%$	K_c	$N_0\%$	K_c	$N_0\%$	K_c
$Y = \sinh(x)$	1024	10,94	1,12	93,75	16,00	96,48	28,44
Geophysical signal	1024	9,38	1,11	38,28	1,62	78,13	4,57

The research results are presented in Table 2. In this table, the values are determined as follows.:

$$K_c = N/(N - N0)$$

N – Number of array;

N0% - Percentage of zero-valued coefficients;

Kc – Compression coefficient.

As can be seen from the table, when performing digital processing of values obtained from analytical functions using piecewise-polynomial bases, it was found that the percentage of coefficients equal to zero is approximately 11% for the piecewise-constant basis, 94% for the piecewise-linear basis, and 96% for the piecewise-quadratic basis. Similarly, when processing real data arrays obtained from geophysical observations, these percentages were determined to be 9%, 38%, and 78%, respectively, for the piecewise-constant, piecewise-linear, and piecewise-quadratic bases. This, in turn, demonstrates that the level of data compression and computational optimization achievable in digital calculations is exceptionally high. Thus, the use of piecewise-polynomial bases not only improves accuracy but also enables more efficient utilization of computational resources.

CONCLUSION

The research results demonstrate that piecewise-polynomial algorithms based on the Haar

orthogonal basis provide high efficiency in digital signal processing. Transformations using the Schauder basis and the application of parabolic splines enable improved accuracy in the determination of spectral coefficients. In particular, for piecewise-quadratic bases, spline-based algorithms not only reduce computational complexity but also preserve higher-order derivatives effectively.

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