European International Journal of Multidisciplinary Research and Management Studies (ISSN: 2750-8587)

Received: 08 May 2026 | Accepted: 02 June 2026 | Published: 30 June 2026

DOI: 10.55640/eijmrms-06-06-16

RESEARCH ARTICLE

Derivations, Automorphisms And Local Derivations of Null-Filiform Leibniz Algebras

Jumayeva O'g'iloy Normurodovna
Navoi State University, Navoi, Uzbekistan

VOLUME: Vol.06 Issue06 2026

PAGE: 94-100

Abstract

The paper is devoted to the study of derivations, automorphisms, local derivations and local automorphisms of the null-filiform Leibniz algebra NFn, which is the unique n-dimensional complex nilpotent Leibniz algebra of maximal nilindex. We give explicit descriptions of the derivation algebra and the automorphism group and show that the derivation algebra is an n-dimensional solvable Lie algebra with an abelian ideal of codimension one. The main result states that a linear map on NFn is a local derivation (respectively, a local automorphism) if and only if its matrix in the natural basis is lower triangular (respectively, lower triangular and invertible). Consequently, for every $n \geq 2$ the algebra NFn admits $n(n-1)/2$ linearly independent local derivations which are not derivations.

KEYWORDS

Leibniz algebra, nilpotent algebra, null-filiform algebra, derivation, inner derivation, automorphism, local derivation, local automorphism, lower central series.

INTRODUCTION

Leibniz algebras were introduced by J.-L. Loday as a non-antisymmetric generalization of Lie algebras [1]. They appeared naturally in the study of periodicity phenomena in algebraic K-theory and in the construction of the Leibniz homology of Lie algebras, which is a noncommutative analogue of the classical homology of Chevalley and Eilenberg [2]. Since then the structure theory of Leibniz algebras has been developed intensively, and many classical results of the theory of Lie algebras, such as the Levi decomposition, Engel's theorem and the conjugacy of Cartan subalgebras, have been extended to the Leibniz setting. A systematic exposition of these results can be found in the monograph [3].

The Tashkent algebraic school has made an essential contribution to this area. In particular, Sh. A. Ayupov and B. A. Omirov initiated the study of nilpotent Leibniz algebras of maximal nilindex and described the so called null-filiform and filiform Leibniz algebras [4, 5]. It turned out that, in contrast to the Lie case, there exists a nilpotent Leibniz algebra whose nilindex equals $n + 1$, where n is its dimension. Such an algebra is unique up to isomorphism, and it is called the null-filiform Leibniz algebra. It is a one-generated algebra and it plays the same role in the Leibniz theory as the model filiform algebra plays in the theory of nilpotent Lie algebras.

Derivations and automorphisms are among the most important invariants of an algebra. They describe the symmetries of the multiplication and are closely connected with the cohomology of the algebra and with its deformations. For a classical treatment in the Lie case we refer to [6]. Another direction that has attracted considerable attention during the last decades is the study of local derivations and local automorphisms. The notion of a local derivation was introduced by R. V. Kadison [7] and, independently, by D. R. Larson and A. R. Sourour [8], who studied these maps on operator algebras. A linear map is a local derivation if at every point it coincides with some derivation, where the derivation may depend on the point. The central question is whether every local derivation is automatically a derivation. For finite-dimensional Lie algebras this problem was investigated by Sh. A. Ayupov and K. K. Kudaybergenov [9], who proved that every local derivation of a semisimple Lie algebra is a derivation and showed that nilpotent Lie algebras admit local derivations which are not derivations. Similar questions for simple Leibniz algebras were solved in [10].

In the present paper we give a complete and self-contained description of derivations, automorphisms, local derivations and local automorphisms of the null-filiform Leibniz algebra. We show that the space of derivations has dimension n and compute its Lie structure, while the space of local derivations is much larger: it coincides with the Lie algebra of all triangular matrices with respect to the natural basis and has dimension $n(n + 1)/2$. A similar result is obtained for local automorphisms. Thus the null-filiform Leibniz algebra gives a transparent example of an algebra for which the local derivation problem has a negative answer in every dimension $n \geq 2$.

The paper is organized as follows. In Section 2 we recall the necessary definitions and the classification of null-filiform Leibniz algebras. Section 3 contains the description of derivations together with the structure of the derivation algebra, and Section 4 is devoted to automorphisms. The main results on local derivations and local automorphisms are proved in Sections 5 and 6. In Section 7 all results are illustrated in the three-dimensional case, and Section 8 contains concluding remarks and open problems.

Preliminaries

Throughout the paper all algebras are considered over the field $\mathbb{C}$ of complex numbers and are finite-dimensional. Most of the results remain valid over any algebraically closed field of characteristic zero.

Definition 1. An algebra L with a bilinear multiplication $[\cdot,\cdot]$ is called a (right) Leibniz algebra if for all $x, y, z \in L$ the Leibniz identity holds:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]. \qquad (1)$$

If, in addition, $[x, x] = 0$ for all $x \in L$, then identity (1) turns into the Jacobi identity, so every Lie algebra is a Leibniz algebra. For any element $z \in L$ the operator of right multiplication $R_z(x) = [x, z]$ is a derivation of L; this is just a reformulation of identity (1).

For a Leibniz algebra L we consider the lower central series

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1. \qquad (2)$$

Definition 2. A Leibniz algebra L is called nilpotent if $L^s = \{0\}$ for some s. The smallest such s is called the index of nilpotency (nilindex) of L. An n-dimensional Leibniz algebra is called null-filiform if $\dim L^k = n + 1 - k$ for $1 \leq k \leq n + 1$.

It is easy to see that the nilindex of an n-dimensional nilpotent Leibniz algebra does not exceed $n + 1$, and null-filiform algebras are exactly the algebras with the maximal possible nilindex. For Lie algebras this value is never attained when $n \geq 2$, because $[x, x] = 0$ forces the algebra to have at least two generators.

Theorem A [4]. Any n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra NF_n with a basis $\{e_1, e_2, \dots, e_n\}$ and the multiplication table

$$[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1, \qquad (3)$$

where all omitted products of basis elements are equal to zero.

For convenience we introduce the linear shift operator S on NF_n defined by $S(e_i) = e_{i+1}$ for $i < n$ and $S(e_n) = 0$. If $x = x_1e_1 + \dots + x_ne_n$ and $y = y_1e_1 + \dots + y_ne_n$, then by (3)

$$[x, y] = y_1S(x). \qquad (4)$$

Lemma 1. The algebra NF_n is a Leibniz algebra, its right multiplication operators satisfy $R_y = y_1S$, and the element e_1

generates NF_n.

Proof. By (4), the product $[y, z]$ belongs to the span of $e_2, \ldots, e_n$, hence its first coordinate is zero and $[x, [y, z]] = 0$. On the other hand, $[[x, y], z] - [[x, z], y] = z_1y_1S^2(x) - y_1z_1S^2(x) = 0$. Thus identity (1) holds. The formula $R_y = y_1S$ is a restatement of (4). Finally, $e_i = S^{i-1}(e_1)$ is obtained from e_1 by repeated right multiplication by e_1, so e_1 generates the algebra. ■

Definition 3. A linear map $d : L \to L$ is called a derivation if $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in L$. A bijective linear map $\varphi : L \to L$ is called an automorphism if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L$. The sets of all derivations and all automorphisms are denoted by $Der(L)$ and $Aut(L)$, respectively.

Definition 4. A linear map $\Delta : L \to L$ is called a local derivation if for every $x \in L$ there exists a derivation $D_x \in Der(L)$, depending on x, such that $\Delta(x) = D_x(x)$. A linear map $\Phi : L \to L$ is called a local automorphism if for every $x \in L$ there exists an automorphism $\varphi_x \in Aut(L)$ such that $\Phi(x) = \varphi_x(x)$.

The sets of local derivations and local automorphisms are denoted by $LocDer(L)$ and $LocAut(L)$. Obviously $Der(L) \subseteq LocDer(L)$ and $Aut(L) \subseteq LocAut(L)$. Note that $LocDer(L)$ is a linear subspace of the space of all linear operators on L.

Derivations of the null-filiform Leibniz algebra

Theorem 1. A linear map $d : NF_n \to NF_n$ is a derivation if and only if there exist complex numbers $a_1, a_2, \ldots, a_n$ such that

$$d(e_i) = i\cdot a_1e_i + a_2e_{i+1} + a_3e_{i+2} + \ldots + a_{n-i+1}e_n, \quad 1 \le i \le n. \quad (5)$$

In particular, $\dim Der(NF_n) = n$.

Proof. Let d be a derivation and write $d(e_1) = a_1e_1 + a_2e_2 + \ldots + a_ne_n$. For every $i < n$ we have $e_{i+1} = [e_i, e_1]$, therefore

$$d(e_{i+1}) = [d(e_i), e_1] + [e_i, d(e_1)] = S(d(e_i)) + a_1e_{i+1}. \quad (6)$$

For $i = 1$ formula (5) is the definition of the numbers a_k. Assume that (5) holds for some $i < n$. Then $S(d(e_i)) = i\cdot a_1e_{i+1} + a_2e_{i+2} + \ldots + a_{n-i}e_n$, and by (6) we obtain $d(e_{i+1}) = (i + 1)a_1e_{i+1} + a_2e_{i+2} + \ldots + a_{n-i}e_n$, which is exactly (5) for $i + 1$. Hence every derivation has the form (5).

Conversely, let d be defined by (5). We must check the derivation rule on all pairs of basis elements. If $j \ge 2$, then $[e_i, e_j] = 0$, and the image $d(e_j)$ has zero first coordinate, because all basis vectors occurring in (5) have indices at least j. Consequently $[d(e_i), e_j] + [e_i, d(e_j)] = 0 = d([e_i, e_j])$. If $j = 1$ and $i < n$, then the required equality coincides with (6), which holds by the computation above. Finally, for $j = 1$ and $i = n$ we get $[d(e_n), e_1] + [e_n, d(e_1)] = n\cdot a_1S(e_n) + a_1S(e_n) = 0 = d([e_n, e_1])$. Since both sides of the derivation rule are bilinear, d is a derivation. The parameters $a_1, \ldots, a_n$ are arbitrary and independent, so $\dim Der(NF_n) = n$. ■

Denote by D_a the derivation given by (5) with the parameter vector $a = (a_1, \ldots, a_n)$. In matrix form, with respect to the basis $e_1, \ldots, e_n$ (the images of basis vectors are written in columns), D_a is a lower triangular matrix with the diagonal $(a_1, 2a_1, \ldots, n\cdot a_1)$ whose k-th subdiagonal is filled with the constant a_{k+1}.

Proposition 1. For any parameter vectors a and b the commutator of derivations is given by

$$[D_a, D_b] = D_c, \quad c_1 = 0, \quad c_k = (k - 1)(a_1b_k - b_1a_k), \quad 2 \le k \le n. \quad (7)$$

Consequently $Der(NF_n)$ is a solvable Lie algebra. Its derived algebra $N = \{D_a : a_1 = 0\}$ is an abelian ideal of dimension $n - 1$, and $Der(NF_n)$ is the semidirect sum of the one-dimensional subalgebra spanned by the diagonal derivation $D_0 = D_{(1, 0, \ldots, 0)}$ and the ideal N.

Proof. Since a derivation is determined by its value on the generator e_1, it suffices to compute $[D_a, D_b](e_1)$. Using (5) we find

$$D_a(D_b(e_1)) = b_1D_a(e_1) + \Sigma_{k\ge 2}\, b_k(k\cdot a_1e_k + \Sigma_{j\ge 2}\, a_je_{j+k-1}). \quad (8)$$

Subtracting the analogous expression with a and b interchanged, the double sums cancel because they are symmetric in the pairs (a_j, b_k) and (a_k, b_j). The coefficient of e_1 is $b_1a_1 - a_1b_1 = 0$, and the coefficient of e_k, $k \ge 2$, equals $b_1a_k - a_1b_k + k(a_1b_k - b_1a_k) = (k - 1)(a_1b_k - b_1a_k)$. This proves (7). If $a_1 = b_1 = 0$, then $c = 0$, so N is abelian. Formula (7) shows that the image of the commutator lies in N, and for $D_k = D_a$ with $a_k = 1$ and all other parameters zero we have $[D_0, D_k] = (k - 1)D_k$, so N is exactly the derived algebra. ■

Corollary 1. The space of inner derivations $Inn(NF_n) = \{R_y :$

$y \in NF_n\}$ is one-dimensional and is spanned by $S = D_{(0, 1, 0, \dots, 0)}$. Hence, for $n \geq 2$, the quotient $Der(NF_n) / Inn(NF_n)$ has dimension $n - 1$.

Proof. By Lemma 1, $R_y = y_1 S$. The operator S satisfies $S(e_1) = e_2$, so it corresponds to the parameter vector $(0, 1, 0, \dots, 0)$ in (5). ■

Corollary 1 shows that the space of outer derivations of NF_n, which can be identified with the first Leibniz cohomology group of NF_n with coefficients in itself, has dimension $n - 1$. Thus even this simplest nilpotent Leibniz algebra possesses a rich family of nontrivial symmetries.

Several further properties of the derivation algebra follow immediately from Theorem 1 and Proposition 1.

Proposition 2. Let $n \geq 2$. Then the following statements hold: (i) the center of the Lie algebra $Der(NF_n)$ is trivial; (ii) a derivation D_a is nilpotent if and only if $a_1 = 0$; (iii) the maximal torus of $Der(NF_n)$ is one-dimensional and is spanned by D_0, so NF_n is not a characteristically nilpotent algebra.

Proof. (i) Let D_a commute with all derivations. Taking b with $b_1 = 0$ and $b_k = 1$ for a fixed $k \geq 2$, we obtain from (7) that $(k - 1)a_1 = 0$, hence $a_1 = 0$. Taking now $b = (1, 0, \dots, 0)$, we get $(k - 1)a_k = 0$ for all $k \geq 2$. Thus $a = 0$. (ii) The matrix of D_a is lower triangular with the diagonal entries $a_1, 2a_1, \dots, n \cdot a_1$. Such a matrix is nilpotent exactly when all these entries vanish, that is, when $a_1 = 0$. (iii) By (ii), every derivation D_a decomposes as $a_1 D_0$ plus a nilpotent derivation, and the nilpotent part lies in the ideal N. Any torus, being a commutative subalgebra of semisimple elements, meets N only in zero, hence its dimension does not exceed $\dim Der(NF_n) - \dim N = 1$. The operator D_0 is diagonal with the eigenvalues $1, 2, \dots, n$, so it spans a torus of dimension one. Since D_0 is not nilpotent, the algebra NF_n is not characteristically nilpotent. ■

The diagonal derivation D_0 defines the natural grading $NF_n = \bigoplus \mathbb{C}e_i$, where e_i has degree i. This grading is compatible with the multiplication, because the product of elements of degrees i and 1 has degree $i + 1$. In this sense NF_n is a naturally graded algebra, and its derivation algebra is generated by the grading derivation D_0 and the positive degree derivations $D_2, \dots, D_n$, where D_k raises the degree by $k - 1$.

Automorphisms of the null-filiform Leibniz algebra

Theorem 2. A linear map $\varphi : NF_n \to NF_n$ is an automorphism if and only if there exist complex numbers $b_1 \neq 0, b_2, \dots, b_n$ such that

$$\varphi(e_i) = b_1^{i-1}(b_1 e_i + b_2 e_{i+1} + \dots + b_{n-i+1} e_n), \quad 1 \leq i \leq n. \qquad (9)$$

In particular, $Aut(NF_n)$ is an n-dimensional solvable algebraic group.

Proof. Let φ be an automorphism and put $\varphi(e_1) = b_1 e_1 + \dots + b_n e_n$. By (4), for every x we have $[x, \varphi(e_1)] = b_1 S(x)$. Applying φ to (3) we obtain $\varphi(e_{i+1}) = b_1 S(\varphi(e_i))$, and by induction $\varphi(e_i) = b_1^{i-1} S^{i-1}(\varphi(e_1))$, which is (9). If $b_1 = 0$, then $\varphi(e_2) = 0$, so φ is not bijective; hence $b_1 \neq 0$.

Conversely, let φ be given by (9) with $b_1 \neq 0$. Its matrix is lower triangular with the diagonal entries $b_1, b_1^2, \dots, b_1^n$, so φ is bijective. For $j \geq 2$ the vector $\varphi(e_j)$ has zero first coordinate, whence $[\varphi(e_i), \varphi(e_j)] = 0 = \varphi([e_i, e_j])$. For $j = 1$ we have $[\varphi(e_i), \varphi(e_1)] = b_1 S(\varphi(e_i)) = b_1^i S^i(\varphi(e_1))$, which equals $\varphi(e_{i+1})$ for $i < n$ and equals zero for $i = n$, since $S^n = 0$. Thus φ preserves all products of basis elements, and by bilinearity it is an automorphism. The group is solvable because it consists of triangular matrices. ■

It is worth noting that the Lie algebra of the algebraic group $Aut(NF_n)$ coincides with $Der(NF_n)$. Indeed, differentiating the one-parameter family (9) with $b_1 = 1 + t \cdot a_1 + o(t)$ and $b_k = t \cdot a_k + o(t)$ at $t = 0$ we obtain exactly formula (5).

Local derivations

Now we turn to the main result of the paper. We shall use the notation V_m for the subspace spanned by $e_m, e_{m+1}, \dots, e_n$; thus $NF_n = V_1 \supset V_2 \supset \dots \supset V_n \supset V_{n+1} = \{0\}$, and $V_m = (NF_n)^m$ by (3). For a nonzero element x we define its height h(x) as the smallest index m such that $x_m \neq 0$. Then $x \in V_m \setminus V_{m+1}$ exactly when $h(x) = m$.

Lemma 2. Let $x \in NF_n$ be a nonzero element of height m. Then $\{D(x) : D \in Der(NF_n)\} = V_m$.

Proof. Since every derivation maps V_i into itself by (5), the inclusion "⊆" is clear. Let $D = D_a$. Using (5) we compute the coordinates of $D(x)$. The coefficient of e_m is $m \cdot x_m a_1$, because the vectors e_{k+i-1} with $k \geq 2$ have indices greater than $i \geq m$. For $j \geq 1$ the coefficient of e_{m+j} equals

$$(m + j)x_{m+j}a_1 + x_m a_{j+1} + x_{m+1}a_j + \dots + x_{m+j-1}a_2. \quad (10)$$

Let $y = y_m e_m + \dots + y_n e_n$ be an arbitrary element of V_m. We look for a such that $D_a(x) = y$. The system of equations obtained from (10) is triangular with respect to the unknowns $a_1, a_2, \dots, a_{n-m+1}$: the unknown a_1 is determined from the first equation $m \cdot x_m a_1 = y_m$, and the unknown a_{j+1} appears in the equation for e_{m+j} with the nonzero coefficient x_m, while the other terms of this equation contain only $a_1, \dots, a_j$. Since $m \neq 0$ and $x_m \neq 0$ in $\mathbb{C}$, the system has a solution. Hence $y = D_a(x)$, and the reverse inclusion is proved. ■

Theorem 3. A linear map $\Delta : NF_n \to NF_n$ is a local derivation if and only if $\Delta(V_m) \subseteq V_m$ for all $m = 1, \dots, n$, that is, if and only if

$$\Delta(e_m) \in \mathrm{span}\{e_m, e_{m+1}, \dots, e_n\}, \quad 1 \leq m \leq n. \quad (11)$$

Therefore $LocDer(NF_n)$ coincides with the Lie algebra t_n of all lower triangular $n \times n$ matrices with respect to the basis $e_1, \dots, e_n$, and $\dim LocDer(NF_n) = n(n + 1)/2$.

Proof. Necessity. Let Δ be a local derivation. For $x = e_m$ there exists a derivation D such that $\Delta(e_m) = D(e_m)$, and by Lemma 2 the vector $D(e_m)$ belongs to V_m. This gives (11).

Sufficiency. Assume that Δ satisfies (11) and let x be a nonzero element of height m. Then $x \in V_m$, and by (11) the vector $\Delta(x) = x_m\Delta(e_m) + \dots + x_n\Delta(e_n)$ belongs to V_m. By Lemma 2 there exists a derivation D_x with $D_x(x) = \Delta(x)$. For $x = 0$ we may take any derivation. Hence Δ is a local derivation. The conditions (11) mean precisely that the matrix of Δ is lower triangular, which gives the last statement. ■

Corollary 2. For every $n \geq 2$ the algebra NF_n admits local derivations which are not derivations. More precisely, $\dim LocDer(NF_n) - \dim Der(NF_n) = n(n - 1)/2$.

For example, the operator Δ defined by $\Delta(e_1) = e_1$ and $\Delta(e_i) = 0$ for $i \geq 2$ is a local derivation by Theorem 3, but it is not a derivation, since by Theorem 1 a derivation with $d(e_1) = e_1$ must satisfy $d(e_2) = 2e_2$. Corollary 2 agrees with the general phenomenon observed for nilpotent Lie algebras in [9]: algebras with a large nilpotent part usually possess many local derivations which are not derivations.

It is instructive to interpret Theorem 3 in terms of characteristic ideals. Recall that an ideal I of an algebra L is called characteristic if $d(I) \subseteq I$ for every derivation d of L.

Proposition 3. The characteristic ideals of NF_n are exactly the subspaces $V_1, V_2, \dots, V_n, V_{n+1} = \{0\}$. Consequently, a linear map on NF_n is a local derivation if and only if it leaves invariant every characteristic ideal of NF_n.

Proof. Each V_m coincides with the term $(NF_n)^m$ of the lower central series and is therefore invariant under all derivations. Conversely, let I be a subspace invariant under all derivations. Since the diagonal derivation D_0 has pairwise distinct eigenvalues $1, 2, \dots, n$ with the eigenvectors $e_1, \dots, e_n$, the subspace I is spanned by some of the basis vectors e_i. Moreover, I is invariant under the derivation S, so $e_i \in I$ implies $e_{i+1} \in I$. If m is the smallest index with $e_m \in I$, we conclude that $I = V_m$. The second statement now follows from Theorem 3. ■

The following table shows how rapidly the gap between derivations and local derivations grows with the dimension of the algebra.

n	**dim Der(NF_n)**	**dim Inn(NF_n)**	**dim LocDer(NF_n)**	**dim LocDer − dim Der**
2	2	1	3	1
3	3	1	6	3
4	4	1	10	6
5	5	1	15	10
6	6	1	21	15
n	n	1	n(n + 1)/2	n(n − 1)/2

Local automorphisms

Lemma 3. Let $x \in NF_n$ be a nonzero element of height m. Then $\{\varphi(x) : \varphi \in Aut(NF_n)\} = \{y \in V_m : y_m \neq 0\}$.

Proof. By (9), every automorphism preserves V_m and the coefficient of e_m in $\varphi(x)$ equals $b_1^m x_m \neq 0$; this proves the inclusion "⊆". For $j \geq 1$ the coefficient of e_{m+j} in $\varphi(x)$ equals

$$x_{m+j} b_1^{m+j} + x_m b_1^{m-1} b_{j+1} + (\text{terms containing only } b_1, \ldots, b_j). \quad (12)$$

Let $y \in V_m$ with $y_m \neq 0$. Since $\mathbb{C}$ is algebraically closed, we can choose b_1 with $b_1^m = y_m/x_m$; clearly $b_1 \neq 0$. Then the unknowns $b_2, \ldots, b_{n-m+1}$ are found successively from (12), because b_{j+1} enters the corresponding equation with the nonzero coefficient $x_m b_1^{m-1}$. The remaining parameters can be chosen arbitrarily. The automorphism φ constructed in this way satisfies $\varphi(x) = y$. ■

Theorem 4. A linear map $\Phi : NF_n \to NF_n$ is a local automorphism if and only if its matrix with respect to the basis $e_1, \ldots, e_n$ is lower triangular with nonzero diagonal entries. Consequently $LocAut(NF_n)$ coincides with the group B_n of all invertible lower triangular matrices and has dimension $n(n + 1)/2$.

Proof. Let Φ be a local automorphism. Applying Lemma 3 to $x = e_m$ we see that $\Phi(e_m) \in V_m$ and the coefficient of e_m in $\Phi(e_m)$ is nonzero. Conversely, let Φ be lower triangular with nonzero diagonal entries $\delta_1, \ldots, \delta_n$, and let x be an element of height m. Then $\Phi(x) \in V_m$ and the coefficient of e_m in $\Phi(x)$ equals $\delta_m x_m \neq 0$. By Lemma 3 there exists an automorphism φ_x such that $\varphi_x(x) = \Phi(x)$. For $x = 0$ we may take the identity map. ■

We remark that the condition that the ground field is algebraically closed is essential in Lemma 3. For instance, over the field of real numbers and for $m = 2$ the coefficient $b_1^2 x_2$ always has the same sign as x_2, so the real analogue of Theorem 4 requires an additional sign condition on the diagonal entries with even indices.

Theorems 3 and 4 are in complete agreement with each other. Indeed, the group B_n of invertible lower triangular matrices is a connected algebraic group whose Lie algebra is the algebra t_n of all lower triangular matrices. Thus the Lie algebra of the group of local automorphisms coincides with the space of local derivations, in the same way as the Lie algebra of $Aut(NF_n)$ coincides with $Der(NF_n)$. It is also worth noting that $LocDer(NF_n)$ is closed under the commutator, although for general algebras the space of local derivations need not be a Lie algebra.

Example: the three-dimensional case

Let us illustrate the obtained results for $n = 3$. The algebra NF_3 is given by the products $[e_1, e_1] = e_2$ and $[e_2, e_1] = e_3$. By Theorem 1 every derivation has the form

$$d(e_1) = a_1e_1 + a_2e_2 + a_3e_3, \quad d(e_2) = 2a_1e_2 + a_2e_3, \quad d(e_3) = 3a_1e_3, \quad (13)$$

so $\dim Der(NF_3) = 3$. By Proposition 1 the nonzero brackets of the basis derivations are $[D_0, D_2] = D_2$ and $[D_0, D_3] = 2D_3$, where $D_2 = S$ and D_3 maps e_1 to e_3 and annihilates e_2 and e_3.

By Theorem 2 every automorphism has the form

$$\varphi(e_1) = b_1e_1 + b_2e_2 + b_3e_3, \quad \varphi(e_2) = b_1^2e_2 + b_1b_2e_3, \quad \varphi(e_3) = b_1^3e_3, \quad b_1 \neq 0. \quad (14)$$

By Theorem 3 a linear map Δ is a local derivation if and only if

$$\Delta(e_1) = c_{11}e_1 + c_{21}e_2 + c_{31}e_3, \quad \Delta(e_2) = c_{22}e_2 + c_{32}e_3, \quad \Delta(e_3) = c_{33}e_3, \quad (15)$$

with arbitrary complex numbers c_{ij}. Hence $\dim LocDer(NF_3) = 6$, while $\dim Der(NF_3) = 3$. The map (15) is a derivation exactly when $c_{22} = 2c_{11}$, $c_{33} = 3c_{11}$ and $c_{32} = c_{21}$. Let us verify directly that, for instance, the map Δ with $c_{22} = 1$ and all other $c_{ij} = 0$ is a local derivation. For $x = x_1e_1 + x_2e_2 + x_3e_3$ we have $\Delta(x) = x_2e_2$. If $x_1 \neq 0$, we take $a_1 = 0$, $a_2 = x_2/x_1$ and $a_3 = -x_2^2/x_1^2$ in (13); then $D(x) = x_2e_2 + (x_1a_3 + x_2a_2)e_3 = x_2e_2$. If $x_1 = 0$ and $x_2 \neq 0$, we take $a_1 = 1/2$, $a_2 = -3x_3/(2x_2)$ and $a_3 = 0$; then $D(x) = x_2e_2 + (x_2a_2 + 3x_3a_1)e_3 = x_2e_2$. If $x_1 = x_2 = 0$, then $\Delta(x) = 0$ and we may take $D = 0$. At the same time Δ is not a derivation, since $c_{22} \neq 2c_{11}$.

Similarly, consider the diagonal operator Φ defined by $\Phi(e_1) = e_1$, $\Phi(e_2) = 2e_2$ and $\Phi(e_3) = e_3$. By Theorem 4 it is a local automorphism, but it is not an automorphism, because by (14) the diagonal entries of an automorphism are b_1, b_1^2, b_1^3, and the equalities $b_1 = 1$ and $b_1^2 = 2$ are incompatible. Let us check the local property directly. Let $x = x_1e_1 + x_2e_2 + x_3e_3$, so that $\Phi(x) = x_1e_1 + 2x_2e_2 + x_3e_3$. If $x_1 \neq 0$, we choose $b_1 = 1$, $b_2 = x_2/x_1$ and $b_3 = -x_2^2/x_1^2$ in (14); a direct computation gives $\varphi(x) = x_1e_1 + (x_2 + x_1b_2)e_2 + (x_3 + x_2b_2 + x_1b_3)e_3 = \Phi(x)$. If $x_1 = 0$ and $x_2 \neq 0$, we choose $b_1 = \sqrt{2}$ and find b_2 from the equation $\sqrt{2}\cdot x_2b_2 + 2\sqrt{2}\cdot x_3 = x_3$; then $\varphi(x) = 2x_2e_2 + x_3e_3 = \Phi(x)$. Finally, if $x_1 = x_2 = 0$, then $\Phi(x) = x$ and we can take the identity automorphism.

CONCLUSION

In this paper we have studied the null-filiform Leibniz algebra NF_n, which is the unique n-dimensional nilpotent Leibniz algebra of maximal nilindex. We have obtained explicit descriptions of its derivations and automorphisms, determined the structure of the Lie algebra $Der(NF_n)$ and showed that the space of outer derivations has dimension $n - 1$. The main results, Theorems 3 and 4, show that local derivations and local automorphisms of NF_n are exactly the triangular operators with respect to the natural basis, which is equivalent to preserving the lower central series. As a consequence, for every $n \geq 2$ the algebra NF_n has $n(n - 1)/2$ linearly independent local derivations which are not derivations.

The method used in the paper is based on the fact that the orbit of an element under the action of derivations or automorphisms is completely determined by its height with respect to the lower central series. It seems natural to apply this approach to other one-generated nilpotent Leibniz algebras, in particular to naturally graded filiform Leibniz algebras [3, 4], and to the study of 2-local derivations and 2-local automorphisms of these algebras. These problems will be considered in forthcoming publications.

The obtained results may also be used in teaching courses on nonassociative algebra. The null-filiform algebra is simple enough to allow complete computations by hand, and at the same time it clearly demonstrates the difference between the global notion of a derivation and its local counterpart. Explicit formulas (5), (9), (11) and the example of Section 7 can serve as a basis for exercises and for computer experiments with more complicated Leibniz algebras.

REFERENCES

1. Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz // L'Enseignement Mathématique. 1993. Vol. 39. P. 269–293.
2. Loday J.-L., Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology // Mathematische Annalen. 1993. Vol. 296. P. 139–158.
3. Ayupov Sh., Omirov B., Rakhimov I. Leibniz Algebras: Structure and Classification. Boca Raton: CRC Press, 2019. 323 p.
4. Ayupov Sh.A., Omirov B.A. On some classes of nilpotent Leibniz algebras // Siberian Mathematical Journal. 2001. Vol. 42, No. 1. P. 15–24.
5. Ayupov Sh.A., Omirov B.A. On Leibniz algebras // Algebra and Operator Theory: Proceedings of the Colloquium in Tashkent, 1997. Dordrecht: Kluwer Academic Publishers, 1998. P. 1–12.
6. Jacobson N. Lie Algebras. New York: Interscience Publishers, 1962. 331 p.
7. Kadison R.V. Local derivations // Journal of Algebra. 1990. Vol. 130. P. 494–509.
8. Larson D.R., Sourour A.R. Local derivations and local automorphisms of B(X) // Proceedings of Symposia in Pure Mathematics. 1990. Vol. 51, Part 2. P. 187–194.
9. Ayupov Sh.A., Kudaybergenov K.K. Local derivations on finite-dimensional Lie algebras // Linear Algebra and its Applications. 2016. Vol. 493. P. 381–398.
10. Ayupov Sh.A., Kudaybergenov K.K., Omirov B.A. Local and 2-local derivations and automorphisms on simple Leibniz algebras // Bulletin of the Malaysian Mathematical Sciences Society. 2020. Vol. 43. P. 2199–2234.